Random Processes (Contd..)

Okay we will continue with our discussion on random processes and see whether we can find it up today that is most of it and then quickly recollect what we did last time we discussed the concept basic concept of a random process and also the concepts of simplifying the characterization of a random process through assumption um stationarity wise sense stationarity and (…) right so I think that’s maybe work lets continue the discussion now from now onwards we will assume that we are closely working with wide sense stationary process right processes which are at least wide sense stationary right they may or may not be strict sense stationary but we can assume that they are at least wide sense stationary and please recollect that as for as wide sensationary is concern you are only required to [noise] worry about two properties namely the mean value function and the auto correlation function the mean value function is maybe the constant independent of time the auto correlation function is going to be a function only of the long variable t one minus t two rather t one and t two okay now therefore if these two functions more or less are I complete certain order characterization of a wide sense stationary process that they are not complete characterization the complete (…) order characterization right the just look at the properties of the auto correlation function is some more detail so for now onwards we will denote the auto correlation function by R tau R sub x tau R sub y tau etcetera so we just looking at the property [noise] some of the I am just going to mention some of the properties because we are very simple to prove um the more or less follow directly from the definition of the auto correlation function or at does a little bit of manipulation and I am sure you already know this I am not spend too much time on it I am just mention them so a real process x t x of t right the auto correlation function could be real if you also real valued right if it’s a complex function um if you process x t is a complex valued function then auto correlation function could be complex value right it will be an even function so even function see already seen last time because if I change the order of t one and t two you multiplying x t one and x t two it will not make any difference to the average value of the product right so whether you write t one minus t two or t two minus t one is the same thing right so its an even function so if I replace tau with minus tau the value will be the same again for complex valued functions the symmetry this even symmetry will change into contiguous symmetry (Refer Slide Time 05:38 min)
so for complex valid functions the \((\ldots)\) function will be contiguous symmetric
what is what is that is Rx minus tau could be equal to Rx continuity tau that is because that
is the communication operation involved in the multiplication then you take the auto correlation
function of two a complex value random process okay
contiguous symmetry is also sometimes for \((\ldots)\) symmetry
and this third property sufficient property number one this is for number two and property
number three is that it is maximum auto correlation function is maximum at tau equal to zero
right
property number four if the random process \(x(t)\) happens to be periodic
so for periodic random processes
the corresponding auto correlation function also will be periodic okay
you have really take for properties to true just animating them for the \((\ldots)\) of computers so that
you are familiar with these things
five this is important very important the fully transform are the auto correlation function for the
fully transform of the auto correlation function which we have seen is nothing but the so called
\((\ldots)\) density function right
and you know a how the define the \((\ldots)\) density function with have expected value of fully
transform of \(x(t)\) magnitudes square right
so take the expected value of some square quantity right
so what can you say about such an expected value it will always be positive right and though for
the fully transform a the auto correlation function of any random process will always be positive
right
it will be real value positive point of view unlike the fully transform um of an arbitrary function
which can even be complex valued function
the free transform of an auto correlation function will always be real valued and positive right
so is non-negative actually note generally the society is non-negative for all frequencies right
and there is one of the important task for let us say checking whether not a given function could
be an auto correlation function of some process are there or not right
we take it fully transform if there is a positive function for all frequencies [noise] then it is it is
likely be not a \((\ldots)\) function of some process of the other
then we how do we know this is subject this is understood by everyone the motivation why this
is this property comes this comes from the definition the parese density function
which we define to be the expected value of some square constant right \(((\ldots))\) must be positive
and we already seen that the parse density function in the auto correlation function are fully
transform \(((\ldots))\) there is by \(((\ldots))\) theorem right
so also the value of \(R_0\) which we also \(((\ldots))\) is maximum preservely for a periodic auto
correlation function the people repeat itself at periodic intervals with period \(t\) whatever is a
period
but in general \(R_0\) is maximum and its value is equal to if have a physical significance is equal
to sigma square
the various have been process at time \(t\) of all time we are suppose we considering white
sentationary processes the various will be the same for all time instance
so its [noise] take which we sometimes also called the total average power [noise] right why
because as you can see if you if you remember \(R_\tau\) would be the inverse weight as some of
\(((\ldots))\) density function right
so if I put \(\tau\) equal to zero what you get \(R_0\) equal to integral of \(Sx\) that is the area under the
density function right and therefore it is a total average power
and the last property there are like to mention here is that um in general for random processes if
you consider the limit limiting value of \(R_\tau\) as \(\tau\) tends to either plus or minus infinitive right
that is you are giving if there is a general needs of the auto correlation function what kind of
function could be the auto correlation function in general as we approach infinitive or either side
plus positive side or negative side
the value of the auto correlation function could tend to this limit will be equal to mew \(x\) square
so \(Rx\) \(\tau\) equal to mew \(x\) square here mew \(x\) is a mean value okay and this is a process happens
to be zero mean then what will happen to this limiting value for either side written to zero
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now this I am not truing but very simple to prove in \(((\ldots))\) even more simpler to \(((\ldots))\) what we
are say suppose its zero mean process just for the \(((\ldots))\)
what we are saying is every auto correlation function for zero mean process it tend to zero as \(\tau\)
tends to infinitive
does it may contiguity sense what does \(Rx\) just remember try to ask yourself what does the value
of \(R_\tau\) represent for a given value of \(\tau\) it represent the cross correlation between two random
variables which are sampled which have team by sampling the random process at two time
instance which are separated by \(\tau\) seconds is in it
it is the correlation between two random variable which are separated by tau seconds and what this says is the larger you make the separation the smaller we correlation between them and it will become zero as tau density infinitive right
that is if the separation between then becomes very large there is every reason to expect that the relative values will have absolutely no correlation respective each other right
so that basically that makes lot of ((...)) random process it will happen that way right please close by ((...)) would have a larger correlation that is why R zero is maximum right and then it starts to ((...)) starts to ((...)) ((...)) to zero for either side okay
so typical form misplace
(Student:Question)
if it is a periodic function then this will not be true right this property is not valid for periodic auto correlation functions it is only when for a periodic participation function

so basically what we are saying is you are right auto correlation function typically you will have such as like that right in the case zero as tau density this is the plot of it become auto correlation function as tau tends to infinitive or minus infinitive it has to be even symmetric if it is um even value for real value process and so on and so forth [noise] okay
at this point having define the basic concepts which characterize random process it is now possible for us to look at the very important specify process which we shall using in modeling noise in communication systems right
and in fact the process this particular random process is known that the name of in white noise process right
so let me define a process what you look kind of process which is called a white noise process
if I am take it up just like into like to reemphasize of fact that the [noise] the auto correlation function and the ((...)) density function ((...)) equivalent certain order descriptions of the random process is in it
because of the free transform if you give one you know the other right
so the more certain order descriptions and secondly these descriptions or at that we are discuss or independent of what kind of density function the process has right
because you are only looking at the first two movements of the density function
the first order movements um it is the mean function and the second order movement which is the auto correlation function
you could have any density function right [noise] and you could have random processes with different density function different joint density functions having the same ((...)) properties right
so just something about keep it the back of your mind the ((...)) discussing only second order properties we are ignoring the density function properties with we are ignoring the detail you only looking at these two cross properties
the detail properties are not known to us or we are not talking about the ((...)) okay

let we compare to the white noise process the white noise process is one is really the definition is very simple for which ((...)) density function as this form
it is constant in this just the value arbitrary value or arbitrary notation for the constant individually denoted by N sub zero by two for all frequencies okay
for all values of the frequency f the auto correlation um ((...)) density function is a constant so the power ((...)) place to function plot as a function of f looks like this
and make density why the name comes from the name comes from effect that all frequency components are present in equal measure
so the analogy comes from white light right we are a large number of frequency components constitute the white light okay so that’s for the name function
if the process is if the (\(\ldots\)) density function is not flat then if you want to call it we can call it a
colored random process the colored noise process
because it is not so it is just as again wide right so it is not flat we call it is sometimes we just
loose the quality colored noise process right
and they are various kinds of colors the one thing have but usually the simply quality color trend
noise process
now what will be the auto correlation function of this process
that is the \(R_x(\tau)\) of white noise process what is \(R_x(\tau)\) it will be delta function will be \(N_0/2\) by two delta \(\tau\) right
because this is the free transform \((\ldots)\) the free transform this equal to this the \([\text{noise}]\) inverse free transform of this is this right
so the auto correlation function therefore plot just like this
this is delta \(\tau\) um sorry this is \(\tau\) and this is \(N_0/2\) by two delta \(\tau\) \(\tau\) equal to zero the delta function occurs \(\tau\) equal to zero
this \(N_0/2\) by two is more or less standard notation using communication theory it essentially
says this is called two \((\ldots)\) density value the value of \(N_0/2\) by two is called two side \((\ldots)\) density
in as much as this negative fitness is only abstract entities the corresponding one side \((\ldots)\) density sometimes called \(N_0\) is denoted by \(N\) the twice of this right
there is a few flock the negative access also the or if you only talking in themselves positive fitness right
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the total power of frequency in a \((\ldots)\) frequency \(f\) right will be actually the some of these two
areas right remember that
because every real \((\ldots)\) with frequency \(f\) will also have \((\ldots)\) minus \(f\)
so really speaking if you want to compute the power contain in a regime to \(f\) to \(f + \Delta f\)
equal to get a this area with this area right
so \(N_0/2\) by two \(\Delta f\) plus \(N_0/2\) by two \(\Delta f\) which will become \(L\) zero \(\Delta f\) right
so $L_0$ is sometimes called single density for the white noise process and $L_0$ by two could become the double density it’s a just general terms which are used in the free now this is a very convenient modify many physical random processes that become a process communication theory that remember it is only a model why because such a process cannot physically exist can you see that why can you give why reason why cannot physically exist that is one way up to concorancity um just look at slightly more slightly different way what is a total power on the this process infinitive right no physical process cannot infinite problem right because it is really infinitive right alternatively if you look at the auto correlation domain what is its say that other than the fact that you know $X_t$ could be relative $X_t$ will be correlative will $X_t$ if you need even slight um in the time in the time remain along with time exist if you slightly and if you look at two samples which are very very close together but not the closes is not equal to zero right they will be uncorrelated right [noise] pershop should be happen persophs wide sense to occur the process must have infinite problem otherwise it will be correlated right otherwise the process will have some smoothness and exist even a small minute degree of smoothness this kind may happen right in an NM time function that we are need right so basically infinite power or such an auto correlation function or idealization which will never occur in practice however is of the this is so this is convenient to model many physical processes in communication theory with this model because the more or less satisfy the broad characteristics that this model assumes for example that kind of a noise is if you deal within communications the thermal noise it’s a very large spectrum very wide spectrum more or less it is flat over the spectrum so it is that is really go to infinitive infinitive is anywhere in a spectrum its goes very wide much larger than the bandwidth of the signal that you are working with so for all practical purposes with modular’s white okay so even though in practice it is not really wide in the true size of the work (Student:Question) um most thermal noise short noise and various other kinds of noise many of them can we modularize wide um but their situations maybe have to be due with the non white model okay their situations so um it doesn’t mean the you will always have real product you always needs to work with only white process right situation when you have to look at the access spectrum of the noise which is typically not wide in some situations right okay at this point um I think its important for us to understand one additional set of relations which typically need to work with many times you will be faced with any analyzing systems analyzing the system in the random inputs in apteral when you work when you our purpose of being this um view right now is to have the necessity tools to analysis the performance of communications system in the process presence of noise
and if you have if you remember whatever we discussed so for most communication system will involve one of the major components of any communication system could be filters of various kinds right you are doing a lot of filtering here and there right so like to know if there is a certain kind of process that is random process which is input your filter what kind of random process within output of this filter right so like to know this relationships so lets look at these relationship that is what happens when you transmit random process to a linear system [noise] right your linear system [noise] or a linear filter so we have a linear system system will invoice or process of t input is random process X of t output is some random process Y [noise] okay so in as much as we are concentrating only on the wide sense stationary processes that is we um equal to assume that x is a white sense stationary process we will also try to characterize Y in terms of first movements of course you will um it can be shown the pie t also will be white sense stationary maybe to be obvious even if it is not it can be show right so that means given the mean value function and the auto correlation function of the input process you like to find out what is the mean value function and the auto correlation function of the of ((…)) there is a concern ((…)) so lets look at the mean value function how is ((…)) related to x t to the conclusion of relation so basically that’s why starting point [noise] so if I look at expected value of Y t that is the mean value function of [no volume] first let me express Y t Y t in terms of h t is x tau x t minus tau eeta right since I think expected value of this basically where are is a expected value of this integral for this convolution evaluation right x t is a deterministic filter x t is a deterministic function an expectation is also linear operator it’s a linear operator with expected density function of the process right and because it’s a linear operator and a certain conditions you can clarify the um interchange these two integral operators the integral operators corresponding to the expectation operation and this integration [noise] right (Refer Slide Time 26:47 min)
so I can carry the um expectation of vector inside the integral right as the expected value of the product of these two but x t is not random so we can keep it like that x tau and really speaking the averaging will operate of the process x t right you are using the linearity property of the expectation of ((…)) so what is this this is ((…)) the mean value function of the this is mew x [noise] t minus tau and since we are assuming wide sense stationarity right what will be the value of mew sigma x star will be constant right so it will be minus tau theta minus infinitive to infinitive h tau into mew x mew x is a constant with comes out of the ((…)) so just what you have right so the mean value at the output is the mean value of the input so that shows that the input process is stationary up to first order the output process also be stationary ((…))
because this is not a function of time right x t E x t not a function of time you find a new variety also lot a function of time right because this is going to be fixed value some number what is that number the multiplying E x with the area and that the implies response can you specific in terms of in terms of frequency domain valuation this is equal to so mew Y is equal to mew X into H zero right is in it because this is nothing but h capital H of zero H of f is H tau ((…)) J two by f tau it put f equal to zero you can this right and that is ((…)) very appealing because what is it you can thing of the mean value of the input as a DC component of input process right linear frequency component mean value is ((…)) you can consider to be the zero frequency component right and so what um what it say with the mean value of the output is a mean value of input multiply by the response the system at DC DC responsible system right so that’s the first solution very simple basic solution so if you know the frequential response in particular if you know the frequential response or f equal to zero that’s how the mean values in the related so this solution should be very simple however the next value is not that simple but doesn’t matter its still not very complicated so lets look at the auto correlation function so we want to pie not R Y t u which high definition is expected value of Y t into Y u now you might have note is that are not done able to little t minus u because I am not true yet that the output process will be stationary right so we start with writing like this and if ((…)) to be a function of only t minus u then it becomes the stationary process right so just look at this [noise] this by definition is um now I just um even just substitute the value of Y t and Y u this is ((…)) mechanical process use the dummy variable tau one x t minus tau one d tau one use a dummy variable tau two its u minus tau two d tau two expected value of right so its fast its substitute from Y of t substituting from Y u is defined dummy variables to differentiate between integrals can write this like this um taking the linear operator um expectation of theta once again this side combining with these two random entities these are the two random entities
so the expectation will now work on the product of this and this
so $X_t$ minus $\tau_1$ one $X$ of $u$ minus $\tau_2$ okay of course certain conditions are required to be satisfied for this to be valid that we will assume the conditions are valid
the basic conditions are that $h$ of $\tau_1$ with corresponding stable system and $x$ of $t$ should be a finite in the reprocess and infinite power process okay these are the conditions which are generally required will not go into details okay
so this is theta one its ($(\ldots)$) I think is a purely mechanical um mediculation [noise] its should not how difficulty
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and what is this quality this expectation this is $R_x t$ minus $\tau_1$ one okay at the even less that may write the general expression $u$ minus $\tau_2$ two if you assume with a stationary process right then it will be function only of this minus this right
so if for WSS of a wide sense stationary $x$ of $t$ let um just define $\tau$ equal to $t$ minus $u$ right
so this becomes can you see the regime of function of $t$ minus $a$ right
because it will be a function this minus this or this minus this [noise] right
so this because [noise] I can write $R_y t$ its separating $R_y t$ $u$ I can now write $R_y t$ tau because it’s a function only $f$ $t$ minus $u$ could be equal to just rewriting this write rewriting this [noise] solution [noise]
you have $h$ $\tau_1$ one I think you can look at a lower check you have extra to what will be write here $R_x$ take the difference it will become $\tau$ minus $\tau_1$ plus $\tau_2$ right $d$ $\tau_1$ one $d$ $\tau_2$ two [noise] okay
so as you can see if a function um this $\tau_1$ one and $\tau_2$ two could be disappear after these [noise] two integral integration gas been done
so you will be left only with the function of $\tau$ [noise] $R_y$ is a function of $\tau$ right
so which means which implies that $Y$ $t$ is also WSS
so if you pass [noise] wide sense stationary random process so a linear timing variant filter right because $X_t$ was a linear time variant filter in fact that is the recent output process is wide sense stationary [noise]
for example if the filter was a time variant filter even though the input process was wide sense stationary output process then need not be wide sense stationary because the fact your filter imposed response keeps value with time right
so for the output process to be wide sense stationary input not only the input process should be wide sense stationary the filter [noise] to which (…) should be timing variant

so timing variants is the clue which we have listed okay
so what you find you find there is some kind of since like this relationship that you have which complicates is but actually it’s a very simple relationship right
I will not go through the (…) I give this an exercise (…) exercise for you to complete because these are it’s a repetition of the same steps arrive done so for right
and that we help you to see the relationship in a slightly better lite [noise]
is no where it the looking at the auto correlation function of the output right there is a system where we are working with what we have done so for is you are related the output auto correlation function with the input auto correlation function go through a two step process
first try to relate the cross correlation between the input output input and the output to the auto correlation function input
that is consider expected value of X t into Y u rather Y t into Y u right then we have to only deal with y integral fran two integrals is in it
I should have define if the concept of cross correlation function which have not done but let me let me complete that
so just like you have the auto correlation function of a process which is Rx t tau equal to Rx t one t two equal to expected value of x t one into x t two I can define a cross correlation function between two processes x and y and two processes x and y in the same error
what we are do I sample the random process X t at the time instance t one sample the random process Y t at a time instance t two have to could to random variables now x t one and y t two take the cross correlation
so that is the definition of cross correlation function of two random processes so it will be x t one into y t two okay
so basically what I am suggesting is first find out for this (…) picture the cross correlation between these two processes the input process and the output process [noise] right
and then find the output auto correlation function in terms of this cross correlation right if you go to this two step process what we just see the result is verify them then this whole relationship becomes much more meaningful
so what have to verifies the following so that Ryx tau is equal to Rx tau convolved with h of minus tau okay one simple relation
and um of course unlike the auto correlation function or the cross correlation function does not have that even symmetric property right
because we have different processes now if you change the order of x t and y t it will never change it will make it will different function right
so I mean (…) relationship but it not be in symmetric relationship example if as a place if I change the order a first x and then take y right then you find again (…) is like this Rx tau convey with h tau so they are equivalent okay
the general relationship between Ryx tau N R x tau Rxy tau could be this you see the difference you are say R ys is tau equal to Ryx minus tau right it is Rx y minus tau right
which is obvious again just look at the definition this things will more or less follow from that [noise] okay
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so this is a one relationship we are able to expose what is that say the cross correlation between the input and the output is the auto correlation function input convolve the impose is passed either directly depending on whether you are having this kind of cross correlation or within mirror image of the imposes process right around the origin right

in the second step we show that we can express [noise] excuse me R yy tau or R y tau usually write simply Ry tau to simplify can we written us R yx tau convered with tau right how will this gives you show this you know you start with expected value of y t into y t plus tau that is the definition of R y tau right they substitute only pie Y t plus tau [noise] keep this here in terms of x t I am go through that process so you will ultimately end up here

so if I combine this solution with a previous solution which has this what you get um if I combine these two relations what will you get R y tau is equal to I am substituting for R yx tau is R x tau convey with X minus tau and this convey with h tau

lets precisely what this relationship is that we defined okay so now if you a look at this relationship in a much simpler manner its really a double convolution double convolution of the auto correlation function of the input with h minus tow first and h tau later or the other variant okay because communication is the competitive operation (Refer Slide Time 40:47 min)
so this relationship with which the rather complicated is actually this which is much more ((…))
(Student:Question)
no no what I am saying is can I why any substitute for x of minus t plus tau [noise] ((…) that is
what we are need to R yx right ((…)) okay
now this is there for the relationship between the input auto correlation function and the output
auto correlation function
incidentally this relationships are very useful
let’s look at these relationship for example right which we can prove very easily which unsure
you will be able to ((…) exactly one step
I will just say look at this relationship this is extremely useful and this um I will discuss
application of this very briefly
this terms be the ((…) cross correlate the input in the output random processes what is am see is
now um this convolution
suppose I choose the input auto correlation function to be a white noise process input process to
be a white noise process what will be the corresponding auto correlation function auto
correlation function
delta tau some cost density delta tau what will be the cross correlation of input and output know
will be simply h tau right
so what is the result if I feel a white noise process at the input to a linear time function the cross
correlation between the input white noise and the output noise which may not be white which
will not be white actually right
could the cross correlation function could be proportional to the imposed this possible system
so this gives me a method of finding the imposed responsible unknown system experimentally
right
I feel white noise at the input to the system look at the output random process cross correlate
these two processes right and the result of cross correlation will identify the imposes
functionality it otherwise not known to me that’s why its not known to you right

so this is the very useful relationship where can I use it in communication systems ((…) right
the typical unknown thing in a communication system is the channel system to which the um
signal is being passed right
so many many physical um many many physical communication system actually use this method
to identify which ((…) process
example your um your GSM standard which we are using the mobile telephonic transmits wide in the form of packets information in the form of packets

every packets has its as in the middle of it a training sequence right which is socially some kind of a (...)) serial noise sequence serial noise white sequence right which is parse um if you look at the output of the corresponding to the training sequence of the receiver and see the most or can find out to cross correlation between what was transmit to what is receive right the um the estimate of the channel is impose is (...) and that is one used to further process to signal to get your nicely okay

so this relationship is extremely useful (...) what I am saying we discuss it further if necessary now finally this illustration implies a corresponding relationship in the frequency domain right lets look at that relationship just take the free transform both the sides so what will be the free transform of this the (...) density function of R y t right

so it transform of five y f will be S y f

this will be S x f convolution in time domain will reduced multiplication in the frequency domain right

so what will be the product here S x f into h of f and will be the free transform of this x conjecute of f right

so what you get H f into H conjecute f it is H f mod square this is the corresponding very simple in (...) appealing frequency the minor relationship okay that is the input the output parse density function is obtained by multiplying the input parse density function by w h square of the transfer function right

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in the sense because parse density function is a specific function you must multiplied by positive number to get make sure the reason always positive that makes a lot of sense

any questions

any questions so far

no okay

finally let we spent some time on if returning to density functions

so far when I was discussing purely in terms of white sense stationarity and ignore the concept of density function because how is looking at a second order characterization of the process right
but it is useful to know which are mentioned before to you that many physical processes the density function happens to be Gaussian right so we need to take a look at these functions or these processes in some more detail so in fact most of the time the kind of noise process we will deal with will model the second order properties by saying that it’s a white process that is the density function is flat but we are leave it at that will also say we say something about the density function we say it’s a white noise it’s a white Gaussian noise so now we not only talking about second order property but we are giving some more detail information that is which is the density function associated with that is some Gaussian you understand the notion of the Gaussian random variable the notion the nation we like to understand is that of a Gaussian process when we say process is Gaussian what is need we know that when a random variable $X$ is Gaussian what is truly needed is the density function is this right so the question is what you mean by same the $X_t$ is a Gaussian process [noise] what is this mean therefore to like to understand a little bit okay to start with let we recap one very important property of a Gaussian of Gaussian random variables suppose $X_1$, $X_2$, $X_n$ all Gaussian random variables okay I will just take care of that suppose we have Gaussian random variables then if I construct a linear combination of these Gaussian random variables by saying that by constructing $Y$ in terms of $X_i$ $g_i$ is some arbitrary coefficient of linear combination I going from one to $n$ right (Refer Slide Time 50:06 min)

![Gaussian Process]

then $Y$ will be the Gaussian right we know this now I am say this is a slightly different way we will defined a set of random variables $X_1$, $X_2$, $X_n$ to be jointly Gaussian this is the definition know you say that $X_1$, $X_2$, $X_n$ or jointly Gaussian such that if I take any linear combination of these the output random variable then the resulting random variable is process so if this is Gaussian for an arbitrary set of coefficients $g_i$ arbitrary twice of $g_i$ right this will form the basis for definition of the Gaussian random process which are take up next thank you very much